Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.
we obtain as their internal product

\[ a \mid b = (a_1 \beta_1 + a_2 \beta_2)(\beta_1 \epsilon_1 - \beta_2 \epsilon_1) \]

\[ = (a_1 \beta_1 + a_2 \beta_2)(\epsilon_1 \epsilon_2) \]

\[ = a_1 \beta_1 + a_2 \beta_2. \]

Similarly we find for the internal square

\[ a \mid a = a_1^2 + a_2^2. \]

[TO BE CONTINUED.]

---

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.

By Mr. Marcus Baker, Washington, D.C.

[CONTINUED FROM VOL. I, PAGE 138.]

GROUP II. PART I.

\[ \Delta = \]

32. \[ \frac{1}{4} \sqrt{8m_c^2(a^2 + b^2 - 2m_c^2)} - (a^2 - b^2)^2 \]

33. \[ \frac{1}{2} h_c(v \sqrt{a^2 - h_c^2} + v \sqrt{b^2 - h_c^2}) \]

34. \[ \frac{1}{2} \beta_c ab \left( \frac{1}{a} + \frac{1}{b} \right) \sqrt{1 - \frac{1}{4} \beta_c^2 \left( \frac{1}{a} + \frac{1}{b} \right)^2} \]

35. \[ \frac{1}{3} \sqrt{4m_b^2m_c^2 - \left[ \frac{9}{4} a^2 - (m_b^2 + m_c^2) \right]^2} = \frac{2}{3} \sqrt{m_b^2m_c^2 - k_a^4} \]

where \( 2k_a^2 = \frac{1}{4} a^2 - (m_b^2 + m_c^2) \), etc.

36. \[ \frac{h_b^3 \sqrt{a^2 - h_b^2} - h_b^3 \sqrt{a^2 - h_a^2}}{2 (h_b^2 - h_a^2)} \]

37. \[ \frac{1}{6} h_a(v \sqrt{4m_b^2 - h_a^2} + \sqrt{4m_c^2 - h_a^2}) \]

38. \[ \frac{1}{2} ah_a. \]

39. \[ \frac{1}{2} \sqrt{ab} h_a h_b = \frac{1}{2} \frac{a + b}{h_a + h_b} \]
\[ \Delta = 40. \quad h_a \sqrt{\left( h_a^2 + \sqrt{m_a^2 - h_a^2} \right) \left( \frac{m_a^2 - h_a^2}{h_a^2 - h_a^2} - 1 \right)} \]

\[ 41. \quad \frac{1}{2} h_a \left( \frac{1}{4} a^2 + m_a^2 - h_a^2 + \alpha \frac{m_a^2 - h_a^2}{h_a^2 - h_a^2} + V \frac{m_a^2 - h_a^2}{h_a^2 - h_a^2} \right) \]
\[ = \frac{1}{2} h_a \left( \frac{1}{4} a^2 + k_a^2 + ak + V \frac{1}{4} a^2 + k_a^2 - ak \right), \text{ where } k_a = m_a^2 - h_a^2, \text{ etc.} \]

\[ 42. \quad \frac{Rh_b h_c}{a} \]

\[ 43. \quad \frac{ab}{4R} \sqrt{2(a^2 + b^2 - 2m_c^2)} \]

\[ 44. \quad h_a \left( \sqrt{b^2 - h_a^2} \pm \sqrt{m_a^2 - h_a^2} \right) \]

\[ 45. \quad \frac{ab}{8R} \left( a \sqrt{4R^2 - b^2} + b \sqrt{4R^2 - a^2} \right) \]

\[ 46. \quad 2R^2 \left[ (f - 2x^2) \sqrt{(g - 2x^2)(1 - g + 2x^2)} + (g - 2x^2) \sqrt{(f - 2x^2)(1 - f + 2x^2)} \right], \]
where \( f = \frac{m_a^2 + 2m_c^2}{3R^2}, \quad g = \frac{2m_a^2 + m_c^2}{3R^2} \), and \( x \) (\( = \sin C \)) is to be found from the equation

\[ x = \sqrt{(f - 2x^2)(1 - g + 2x^2) + \sqrt{(g - 2x^2)(1 - f + 2x^2)}.} \]

\[ 47. \quad \frac{sh_a r_a}{2(h_a + r_a)} \]

*If \( \mu \) and \( \phi \) are auxiliaries determined from the equations \( \cos \mu = \frac{h_a}{m_a} \) and \( \cos \phi = \frac{h_a}{\beta_a} \), then

\[ \Delta = \frac{h_a^2}{\cos \mu} \sqrt{\frac{\sin 2(\mu - \phi)}{\sin 2\phi}}. \]

[For this elegant expression I am indebted to my friend Mr. Charles H. Kummell, of the U. S. Coast and Geodetic Survey.]

\[ \dagger \sqrt{R^2 h_a^2 - \Delta^2} + \sqrt{R^2 h_b^2 - \Delta^2} = \frac{1}{2} \left( \frac{R h_a h_b}{\Delta} \right)^2 \]
GROUP II. PART II.

\[ A = \]

48. \( \frac{1}{2}ab \sin C \)

49. \( \frac{1}{2} \frac{h_a h_b}{\sin C} \)

50. \( \frac{\frac{1}{2}a^2 \sin B \sin C}{\sin A} = \frac{\frac{1}{2}a^2}{\cot B + \cot C} \)

51. \( \frac{\frac{1}{2}h_a^2}{\sin B \sin C} = \frac{\frac{1}{2}h_a^2 (\cot B + \cot C)}{\frac{1}{2}h_a^2 (\sin 2B + \sin 2C)} \)

52. \( \frac{\frac{1}{2}(a^2 - b^2) \sin A \sin B}{\sin (A - B)} = \frac{\frac{1}{2}(a^2 \sin 2B + b^2 \sin 2A)}{\cot A - \cot B} \)

53. \( \frac{\frac{1}{2}(h_a^2 - h_b^2) \sin A \sin B}{\sin (A - B)} \cdot \frac{1}{\sin^2 C} = \frac{\frac{1}{2}h_a h_b \tan \frac{1}{2}A \tan B \tan \frac{1}{2}C}{\cos A} \)

\[ = h_a h_b \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C \]

\[ \frac{2m_a^2 \sin A \sin B \sin C}{2 \left( \sin^2 A + \sin^2 B + \sin^2 C \right) - 3 \sin^2 A} \]

55. \( \frac{2 \left( m_b^2 - m_a^2 \right)}{3 (\cot A - \cot B)} \)

56. \( \frac{1}{3} \sqrt[4]{4m_b^2 m_c^2 - \left[ 9R^2 \sin^2 A - (m_b^2 + m_c^2) \right]^2} \)

57. \( \frac{2R^4}{m_a^2} \sin A \sin B \sin C \left( - \sin^2 A + 2 \sin^2 B + 2 \sin^2 C \right) \)

58. \( \frac{4m_a^2 + 3a^2}{8 (\cot A + \cot B + \cot C)} \)

59. \( \frac{4 \left( m_a^2 + m_b^2 \right) - 3c^2}{4 (\cot A + \cot B + \cot C)} \)

60. \( \frac{\frac{1}{2}R_a^2}{\sin B \sin C} \sin (B + \frac{1}{2}A) \sin (A + \frac{1}{2}B) \)

61. \( R h_a \sin A \)

62. \( \frac{1}{2}a \beta_a \sin (C + \frac{1}{2}A) \)

63. \( R^2 \sin 2A \left( 1 + \cos C \right) \)

64. \( Ra \sin B \sin C = Ra \left( \cos A + \cos B \cos C \right) \)

65. \( R h_a \tan A \sin B \tan \frac{1}{2}C \)

66. \( R \beta_a \tan A \sin B \tan \frac{1}{2}C \sin (C + \frac{1}{2}A) \)
GROUP II. PART II.—Continued.

\[ J = \]

67. \( \frac{h_a (h_b + h_c)}{4 \sin (C + \frac{1}{2}A) \cos \frac{1}{2}A} \)

68. \( \frac{\beta_a (h_b + h_c)}{4 \cos \frac{1}{2}A} \)

69. \( \frac{1}{4} \beta_a^2 (b + c) \left( \frac{1}{b} + \frac{1}{c} \right) \tan \frac{1}{2}A. \)

70. \( \frac{1}{2} \beta_a \beta_b \sin (B + \frac{1}{2}A) \sin (A + \frac{1}{2}B) \sin C \)

71. \( \beta_a \beta_b \beta_c \left( -\beta_a \sin \frac{1}{2}A + \beta_b \sin \frac{1}{2}B + \beta_c \sin \frac{1}{2}C \right) \)

72. \( \frac{1}{2} a \sin B (a \cos B + \sqrt{b^2 - a^2 \sin^2 B}) = \frac{1}{2} b \sin A (b \cos A + \sqrt{a^2 - b^2 \sin^2 A}) \)

GROUP III. PART I.

73. \( sr = (s-a) r_a \)

74. \( \frac{1}{2} r (\sqrt{2m_a^2 + 2m_b^2 - m_c^2} + \sqrt{2m_a^2 + 2m_c^2 - m_b^2} + \sqrt{2m_b^2 + 2m_c^2 - m_a^2} + \sqrt{2m_a^2 + 2m_b^2 - m_c^2}) \)

75. \( \sqrt{\frac{rr_c}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}} = \sqrt{\frac{r_a r_c}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}} \)

76. \( R \left( h_a h_b + h_b h_c + h_c h_a \right) = \frac{R}{2(s-a)} (h_a h_b - h_b h_c + h_c h_a) \)

77. \( \sqrt{\frac{1}{2} R r (h_a h_b + h_b h_c + h_c h_a)} = \sqrt{\frac{1}{2} R r_a (h_a h_b - h_b h_c + h_c h_a)} \)

78. \( s = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{s-a}{-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} \)

79. \( \sqrt{\left( \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) \left( \frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c} \right) \left( \frac{1}{h_a} \frac{1}{h_b} - \frac{1}{h_c} \right)} \)

\( = \sqrt{\left( \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) \left( \frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c} \right) \left( \frac{1}{h_a} \frac{1}{h_b} - \frac{1}{h_c} \right)} \)
GROUP III. PART II.

80. \( a ( - \beta_a \sin \frac{1}{2}A + \beta_b \sin \frac{1}{2}B + \beta_c \sin \frac{1}{2}C) \)
\( = 2s(\beta_a \sin \frac{1}{2}A + \beta_b \sin \frac{1}{2}B + \beta_c \sin \frac{1}{2}C) \)

81. \( \sqrt{2r_a^2 \beta_a \beta_b \beta_c} \left[ \frac{\cos (A - B) + \cos (B - C) + \cos (C - A) + 1}{\cos A + \cos B + \cos C - 1} \right] \)
\( = \sqrt{2r_a^2 \beta_a \beta_b \beta_c} \left[ \frac{\cos (A - B) + \cos (B - C) + \cos (C - A) + 1}{-\cos A + \cos B + \cos C + 1} \right] \)

82. \( s^2 \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C = (s - a)^2 \tan \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C \)

83. \( r^2 \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C = r_a^2 \cot \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C \)

GROUP IV. PART I.

84. \( a \frac{rr_a}{r_a - r} = a \frac{r_b r_c}{r_b + r_c} \)

85. \( rr_a \frac{r_b + r_c}{a} = r_b r_c \frac{r_a - r}{a} \)

86. \( (a + b) \frac{rr_c}{r + r_c} = (a - b) \frac{r_a r_b}{r_a - r_b} \)

87. \( rr_a \frac{r_b - r_c}{b - c} = r_b r_c \frac{r + r_a}{b + c} \)

88. \( rr_a \sqrt{\frac{r_b + r_c}{r_a - r}} = r_b r_c \sqrt{\frac{r_a - r}{r_b + r_c}} \)

89. \( rr_a \sqrt{\frac{4R - (r_a - r)}{r_a - r}} = r_b r_c \sqrt{\frac{4R - (r_b + r_c)}{r_b + r_c}} \)

90. \( \frac{sh_a r_a}{h_a + 2r_a} = \frac{(s - a) h_a r}{h_a - 2r} \)

91. \( r \sqrt{\frac{h_a r_b r_c}{h_a - 2r}} = r_a \sqrt{\frac{h_a r_b r_c}{h_a + 2r_a}} \)

92. \( \sqrt{\frac{1}{h_a} \left( \frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h} \right)} \left( \frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c} \right) \)
\( = \sqrt{\frac{r_b r_c}{h_a + \frac{1}{h_b} + \frac{1}{h_c}}} \left( -\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) \)
GROUP IV. PART II.

\[ J = \]

93. \( rr_a \cot \frac{1}{2}A = r_r \tan \frac{1}{2}A \)

GROUP V.

94. \( r^2 \cot \frac{1}{2}A + 2Rr \sin A = r_a^2 \cot \frac{1}{2}A - 2Rr \sin A = r_b^2 \cot \frac{1}{2}A - 2Rr \sin A = r_c^2 \cot \frac{1}{2}A - 2Rr \sin A. \)

MISCELLANEOUS EXPRESSIONS FOR THE AREA OF A PLANE TRIANGLE.

If we designate the distances from the orthocentre to the sides of the triangle by \( k_a, k_b, k_c \) and from the orthocentre to the vertices by \( k_a', k_b', k_c' \), then

95. \( J = \frac{1}{2} (ak_a + bk_b + ck_c) \)

96. \( J = \frac{1}{2} (ak_a' + bk_b' + ck_c') \)

97. \( J = \frac{1}{8R} (a^2k_a' \csc A + b^2k_b' \csc B + c^2k_c' \csc C). \)

If we designate the distances between the centres of the escribed circles by \( d_a, d_b, d_c \), then

98. \( J = \frac{1}{4} \sqrt{2d_a d_b d_c r} \sin A \sin B \sin C \)

99. \( J = \frac{d_a d_b d_c r}{16R^2} \)

100. \( J = \frac{d_a d_b d_c}{4R} \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \)

101. \( J = r_a r_b \sqrt{\frac{d_c^2}{(r_a + r_b)^2} - 1}. \)

If \( x_a y_a, x_b y_b, x_c y_c \) designate the rectangular co-ordinates of a triangle in a plane, then

102. \( J = \frac{1}{2} [y_a (x_c - x_b) + y_b (x_a - x_c) + y_c (x_b - x_a)] \)

and, in polar co-ordinates,

103. \( J = \frac{1}{2} [r_a r_b \sin (\theta' - \theta') + r_b r_c \sin (\theta''' - \theta'') + r_c r_a \sin (\theta' - \theta''')] \).

If the equations of the sides of a triangle are

\[ a_x x + b_y y + c_z = 0; \ a_y y + b_x x + c_z = 0; \ a_z x + b_y y + c_z = 0, \]

then

104. \( J = \frac{1}{2} \left[ \frac{[a_x (b_y c_z - c_z b_y) + a_y (b_z c_x - c_x b_z) + a_z (b_x c_y - c_y b_x)]^2}{(a_x b_y - b_y a_x)(a_y b_z - b_z a_y)(a_z b_x - b_x a_z)} \right]. \)


If the co-ordinates of the vertices of a triangle are \( x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c \), and

\[
\alpha = x_a^2 + y_a^2 + z_a^2, \quad \lambda = x_b x_c + y_b y_c + z_b z_c,
\]
\[
\beta = x_b^2 + y_b^2 + z_b^2, \quad \mu = x_c x_a + y_c y_a + z_c z_a,
\]
\[
\gamma = x_c^2 + y_c^2 + z_c^2, \quad \nu = x_a x_b + y_a y_b + z_a z_b;
\]

then

\[105.* \quad 4\Delta^2 = \begin{vmatrix}
  0 & 1 & 1 & 1 \\
  1 & \alpha & \nu & \mu \\
  1 & \nu & \beta & \lambda \\
  1 & \mu & \lambda & \gamma
\end{vmatrix}\]

or, without determinants,

\[
\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4,
\]

where

\[
a^2 = (x_b - x_c)^2 + (y_b - y_c)^2 + (z_b - z_c)^2,
\]
\[
b^2 = (x_c - x_a)^2 + (y_c - y_a)^2 + (z_c - z_a)^2,
\]
\[
c^2 = (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2,
\]

and if \( a, \beta, \gamma, \lambda, \mu \) and \( \nu \), have the values set down above, then

\[
a^2 = \beta + \gamma - 2\lambda,
\]
\[
b^2 = \gamma + \alpha - 2\mu,
\]
\[
c^2 = \alpha + \beta - 2\nu,
\]

and

\[
\Delta^2 = 2(a + \beta - 2\nu)(\beta + \gamma - 2\lambda) - (a + \beta - 2\nu)^2
\]
\[
\quad + 2(\beta + \gamma - 2\lambda)(\gamma + \alpha - 2\mu) - (\beta + \gamma - 2\lambda)^2
\]
\[
\quad + 2(\gamma + \alpha - 2\mu)(\alpha + \beta - 2\nu) - (\gamma + \alpha - 2\mu)^2,
\]

which reduces to

\[
\Delta^2 = 4(\lambda + \mu + \nu)^2 - 8(\alpha \lambda + \beta \mu + \gamma \nu) + 4(a \beta + \beta \gamma + \gamma \alpha).
\]

106. \( \Delta = \frac{1}{8R} \left[ (s - a)D_a^2 + (s - b)D_b^2 + (s - c)D_c^2 - sD^2 \right] \) where \( D, D_a, D_b, D_c \)

\( D_a \) are the distances between the circumcentre and the in- and e-centres.

107. \( \Delta = 4K \), where \( K \) is the area of the triangle formed by joining the feet of the medians.

108. \( A = \frac{L}{2 \cos A \cos B \cos C} \), where \( L \) is the area of the pedal triangle or triangle formed by joining the feet of the perpendiculars.

109. \( A = M \frac{(a + b)(b + c)(c + a)}{2abc} \), where \( M \) is the area of the triangle formed by joining the feet of the internal bisectors.

110. \( A = \frac{2R}{r} \cdot N \), where \( N \) is the area of the triangle formed by joining the points of tangency of the in-circle.

**Concluding Note.**—Owing to some changes and additions in the foregoing article made since its original preparation and while in press, the statement on page 135, Vol. I, that “the total number of formulæ * * * in this collection is ninety-three,” etc., is no longer true.

The number of formulæ in the collection is shown below where the second column shows the number that result from counting as distinct those formulæ arising from permutation.

<table>
<thead>
<tr>
<th>Group</th>
<th>I</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group I</td>
<td>31</td>
<td>31</td>
</tr>
<tr>
<td>Group II</td>
<td>41</td>
<td>123</td>
</tr>
<tr>
<td>Group III</td>
<td>11</td>
<td>44</td>
</tr>
<tr>
<td>Group IV</td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>Group V</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>Miscellaneous</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>110</td>
<td>288</td>
</tr>
</tbody>
</table>

[In the original manuscript all the forms arising from permuting the letters were given in full. To save space in printing, those forms which arose from mere cyclical permutation were omitted.—O. S.]